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STOCHASTIC BEHAVIOR OF
DIGITAL COMBINATIONAL CIRCUITS

by

David K. Cheng
A. V. Shanker

Scientific Report

Grant No. AFOSR-75-2809
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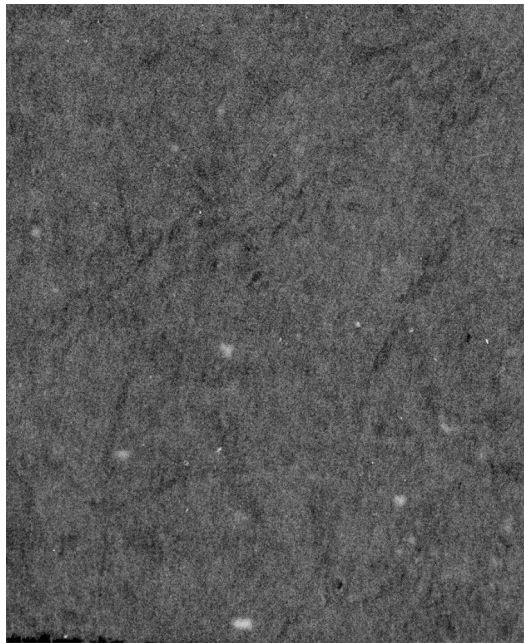
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Department of Electrical and Computer Engineering
Syracuse University
Syracuse, New York 13210

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STOCHASTIC BEHAVIOR OF
DIGITAL COMBINATIONAL CIRCUITS

David K. Cheng and A. U. Shankar
Electrical and Computer Engineering Department
Syracuse University, Syracuse, NY

ABSTRACT

The stochastic behavior of digital combinational circuits is analyzed by the use of Walsh functions. An n -input Boolean function is represented as a Walsh series and the error caused by noise is measured in terms of a distance which is the fraction of the time that the system output due to noise-corrupted signal differs from that due to signal alone. It is shown that the error can be expressed as the sum of two parts: one part depends only on noise statistics, and the other on both signal and noise. Some interesting properties of both parts are discussed and typical examples are given.

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INTRODUCTION

The output behavior of linear (and some nonlinear) analog systems in response to stochastic inputs can be determined in a large number of cases. However, no such comparable facility exists for digital systems such as those employing threshold-type devices and discrete-valued waveforms. Given input statistics and digital system transformations, it is, in general, a formidable task to obtain the output statistics. An obvious reason for this difficulty is that digital device (such as a TTL gate) models are not linear RLC models and cannot be handled by the usual linear algebra and calculus. Logical algebra and an associated calculus that is digital in nature are required [1]. In this connection, Walsh functions have been found to be relevant [2]-[5]. This paper expresses an n -input Boolean function as a Walsh series and defines the error at the output of a combinational circuit caused by noise as the distance between the responses to the signal and to the noise-corrupted signal inputs. The use of Walsh functions appears natural here and does, in fact, facilitate the computation of the system error [6],[7].

It will be shown that the error can be obtained as the sum of two parts: one part depends on noise statistics only, and the other depends on the characteristics of both signal and noise. For independent and identically distributed noise processes, the first part is invariant over each of certain equivalence classes of Boolean functions. Under certain conditions of noise and signal component processes, the system error is expressible as a polynomial function of the expected values of signal and of noise. Some interesting properties of the error polynomial will

be discussed. The Boolean functions of two combinational circuits are studied, and the system errors are computed and plotted for various statistics of signal and noise.

PRELIMINARY CONSIDERATIONS

Any n -input combinational circuit, S , can be represented by a Boolean function that maps every binary n -tuple to either 0 or 1. Let the set of all binary n -tuples be denoted by V_n . Then

$$V_n = \{ \langle x_{n-1}, x_{n-2}, \dots, x_0 \rangle : x_i \in \{0,1\} \text{ for } 0 \leq i < n \} \quad (1)$$

Since an n -tuple can be considered as a vector, it is convenient to denote $\langle x_{n-1}, x_{n-2}, \dots, x_0 \rangle$ by \bar{x} . Let x stand for the integer whose binary representation is \bar{x} ; we have

$$x = \sum_{j=0}^{n-1} x_j 2^j \quad (2)$$

Equation (2) establishes an equivalence between $\{0,1,2,\dots,2^n-1\}$, denoted by D_n , and V_n in Eq. (1). Any function or operation defined on V_n is thus also defined on D_n . Figure 1 shows a schematic diagram of an n -input combinational circuit with a single binary output.

Dyadic operations are fundamental to Walsh functions and their applications. Dyadic addition, denoted by \oplus (on V_n , and hence on D_n), is defined as follows. For any \bar{x} and \bar{y} in V_n ,

$$\bar{x} \oplus \bar{y} = \langle x_{n-1} \oplus y_{n-1}, x_{n-2} \oplus y_{n-2}, \dots, x_0 \oplus y_0 \rangle \in V_n \quad (3)$$

where

$$0 \oplus 0 = 1 \oplus 1 = 0 \quad (3a)$$

and

$$0 \oplus 1 = 1 \oplus 0 = 1 \quad (3b)$$

In fact, (V_n, \oplus) forms the dyadic group for which Walsh functions are the character functions [8].

For an input $\bar{X}(t) = \{X_{n-1}(t), X_{n-2}(t), \dots, X_0(t)\}$: for $t \in T$, $X_i(t) \in \{0,1\}$, for $0 \leq i < n$, where each $X_i(t)$ is a binary stochastic process, the output of a given combinational circuit S is also a binary stochastic process $\{S(\bar{X}(t)) \in \{0,1\}: t \in T\}$. We define a distance $D_S[\bar{X}(t), \bar{Y}(t)]$ between the responses to two inputs $\bar{X}(t)$ and $\bar{Y}(t)$ as the fraction of the time that the output $S(\bar{X}(t))$ differs from $S(\bar{Y}(t))$. If $\bar{X}(t)$ is the signal input process in the absence of noise and $\bar{X}_N(t) = \{X_{N,n-1}(t), X_{N,n-2}(t), \dots, X_{N,0}(t)\} \in V_n: t \in T$ a noise-corrupted input process, then, in general, $S(\bar{X}(t)) \neq S(\bar{X}_N(t))$ and we write the error at the output due to noise at the input as

$$\xi_S = D_S[\bar{X}(t), \bar{X}_N(t)] \quad (4)$$

A more precise definition for ξ_S will be given later.

We assume that the noise is also a binary process $\bar{N}(t) = \{N_{n-1}(t), N_{n-2}(t), \dots, N_0(t)\} \in V_n: t \in T$ and is added to $\bar{X}(t)$ dyadically to yield

$$\bar{X}_N(t) = \bar{X}(t) \oplus \bar{N}(t) \quad (5)$$

The dyadic addition in Eq. (5) is appropriate inasmuch as a combined input digit is in error if and only if the corresponding noise digit is 1. Consequently Walsh functions can be used to advantage in noise-error determination.

Walsh functions, denoted by $Wal(\cdot)$, form an orthogonal basis on

V_n [8], [9]. For $i \in D_n$ and $\bar{x} \in V_n$, the Hadamard-ordered Walsh functions [10], [11], are:

$$\text{Wal}(i, \bar{x}) = (-1)^{\sum_{j=0}^{n-1} i_j x_j} \quad (6)$$

where the notation conforms with that in Eq. (2). Obviously, $\text{Wal}(i, \bar{x}) \in \{1, -1\}$.

The following properties can be readily verified [8], [12]:

$$(a) \quad \text{Wal}(i, x) = \text{Wal}(x, i) \quad (7)$$

$$(b) \quad \sum_{\bar{x} \in V_n} \text{Wal}(i, \bar{x}) \text{Wal}(j, \bar{x}) = 2^n \delta_{ij} \quad (8)$$

where δ_{ij} is the Kronecker delta.

$$(c) \quad \text{Wal}(i, \bar{x} \oplus \bar{y}) = \text{Wal}(i, \bar{x}) \text{Wal}(i, \bar{y}) \quad (9)$$

EQUIVALENT BOOLEAN FUNCTIONS AND THEIR WALSH REPRESENTATION

Because Walsh functions form an orthogonal basis on V_n , we can represent any Boolean function as a Walsh series. We write, for any n -input Boolean function S ,

$$S(\bar{x}) = \sum_{i \in D_n} b_i \text{Wal}(i, \bar{x}) \quad (10)$$

where

$$b_i = 2^{-n} \sum_{\bar{x} \in V_n} S(\bar{x}) \text{Wal}(i, \bar{x}) \quad (11)$$

Two Boolean functions $S_1(\bar{x})$ and $S_2(\bar{x})$ on V_n are said to be equivalent [13] if there is a sequence of permutations and complementations of some of the variables $\langle x_{n-1}, x_{n-2}, \dots, x_0 \rangle$ to produce $\langle y_{n-1}, y_{n-2}, \dots, y_0 \rangle$ such that for every $\bar{x} \in V_n$,

$$S_1(\bar{y}) \oplus S_2(\bar{x}) = d, \quad \text{for } d \in \{0, 1\} \quad (12)$$

Assertion I (See Appendix I for proof.)

Let an n -input Boolean function $S(\bar{x})$ as represented in Eq. (10) be transformed under an equivalence operation (permutations or complementations) to

$$Q(\bar{x}) = \sum_{i \in D_n} q_i \text{Wal}(i, \bar{x}) \quad (13)$$

(a) If $Q(\bar{x})$ is the complement of $S(\bar{x})$, then

$$q_i = \begin{cases} 1 - b_0, & \text{for } i = 0 \\ -b_i, & \text{for } i \in D_n - \{0\} = \{1, 2, \dots, 2^n - 1\} \end{cases} \quad (14)$$

(b) If out of $\bar{x} \in V_n$, the inputs $x_{j_1}, x_{j_2}, \dots, x_{j_m}$ where $m \leq n$, are complemented, then

$$q_i = b_i \text{Wal}(i, h) \quad (15)$$

where

$$h = \sum_{k=1}^m 2^{j_k} \quad (15a)$$

(c) If x_α and x_β ($0 \leq \alpha, \beta < n$) are interchanged, then

$$q_i = \begin{cases} b_i, & \text{for those } i \text{ where } i_\alpha = i_\beta \\ b_i \oplus 2^\alpha \oplus 2^\beta, & \text{for those } i \text{ where } i_\alpha \neq i_\beta \end{cases} \quad (16)$$

This assertion will be used to prove that the part of the error which depends only on noise statistics is invariant over complementations in Boolean functions and over an interchange of inputs under certain conditions.

NOISE ERROR IN COMBINATIONAL CIRCUIT

We have defined previously a distance $D_S[\bar{X}(t), \bar{Y}(t)]$ between the responses of a combinational circuit S to the inputs $\bar{X}(t)$ and $\bar{Y}(t)$ in a

qualitative manner. For stochastic inputs over a discrete time domain T ,

$$D_S[\bar{X}(t), \bar{Y}(t)] = E \left\{ \frac{1}{|T|} \sum_{t \in T} [S(\bar{X}(t)) \oplus S(\bar{Y}(t))] \right\} \quad (17)$$

where $D_S[...]$ satisfies the following properties:

$$(a) \quad D_S[\bar{X}(t), \bar{Y}(t)] = D_S[\bar{Y}(t), \bar{X}(t)] \geq 0 \quad (18)$$

$$(b) \quad D_S[\bar{X}(t), \bar{X}(t)] = 0 \quad (19)$$

$$(c) \quad D_S[\bar{X}(t), \bar{Y}(t)] \leq D_S[\bar{X}(t), \bar{Z}(t)] + D_S[\bar{Z}(t), \bar{Y}(t)] \quad (20)$$

Combining Eqs. (4), (17) and (10), we have the following Walsh representation for the error ξ_S at the output of the combinational circuit S for signal $\bar{X}(t)$ corrupted by noise $\bar{N}(t)$.

$$\xi_S = E \left\{ \frac{1}{|T|} \sum_{t \in T} \left[\sum_{i \in D_n} b_i \text{Wal}(i, \bar{X}(t)) - \sum_{i \in D_n} b_i \text{Wal}(i, \bar{X}_N(t)) \right]^2 \right\} \quad (21)$$

where the identity $a \oplus b = (a - b)^2$ for $a, b \in \{0, 1\}$ has been used. The following relations hold:

$$E\{[1 - \text{Wal}(i, \bar{N}(t))]^2\} = 2 \{1 - E[\text{Wal}(i, \bar{N}(t))]\} \quad (22)$$

which vanishes if $i = 0$; and

$$E\{[1 - \text{Wal}(i, \bar{N}(t))] [1 - \text{Wal}(j, \bar{N}(t))]\} = 0 \quad (23)$$

if i or $j = 0$. By using Eqs. (5), (22) and (23), we can put Eq. (21) in the following form:

$$\xi_S = \xi_S(N) + \xi_S(N, X) \quad (24)$$

where

$$\xi_S(N) = \frac{2}{|T|} \sum_{t \in T} \sum_{i \in D_n} b_i^2 [1 - E\{\text{Wal}(i, \bar{N}(t))\}] \quad (25)$$

which depends only on noise, and

$$\xi_S(N, X) = \frac{1}{|T|} \sum_{t \in T} \sum_{i \neq j: i, j \in D_n} b_i b_j E\{[1 - \text{Wal}(i, \bar{N}(t))] \cdot [1 - \text{Wal}(j, \bar{N}(t))] \cdot \text{Wal}(i \oplus j, X(t))\} \quad (26)$$

which depends on both signal and noise. Note that D_n may be replaced by $D_n - \{0\}$. Equations (24)-(26) are general results for system errors.

Under the complementation operations in Assertion I: (a), (b), which transform S to Q, Eqs. (14), (15) indicate that $q_i^2 = b_i^2$ for $i \neq 0$. Hence $\xi_Q(N) = \xi_S(N)$. Furthermore, using Eqs. (16) and (25) we have

$$\begin{aligned} \xi_Q(N) &= \frac{2}{|T|} \sum_{t \in T} \sum_{i_\alpha = i_\beta} b_i^2 [1 - E\{\text{Wal}(i, \bar{N}(t))\}] \\ &+ \frac{2}{|T|} \sum_{t \in T} \sum_{i_\alpha \neq i_\beta} b_i^2 [1 - E\{\text{Wal}(i \oplus 2^\alpha \oplus 2^\beta, N(t))\}] \end{aligned} \quad (27)$$

Now, in view of Eq. (9), if $N_\alpha(t)$, $N_\beta(t)$ are identically distributed and independent from the other noise processes,

$$E\{\text{Wal}[i \oplus 2^\alpha \oplus 2^\beta, \bar{N}(t)]\} = E\{\text{Wal}[i, \bar{N}(t)]\} \cdot E\{(-1)^{i_\alpha N_\alpha(t) + i_\beta N_\beta(t)}\} \quad (28)$$

Since exactly one out of $\{i_\alpha, i_\beta\}$ will be 1 if $i_\alpha \neq i_\beta$, and since

$(-1)^{N_\alpha(t)}$ and $(-1)^{N_\beta(t)}$ have the same mean, it follows from Eqs. (25) and (27) again that $\xi_Q(N) = \xi_S(N)$. Hence, we can make the following assertion.

Assertion II

For a given combinational circuit S, the noise-dependent error term

$\xi_S(N)$ is

(a) invariant under a complementation of its Boolean function and/or a complementation of any subset of the inputs, and

(b) invariant under an interchange of inputs $x_\alpha(t)$ and $x_\beta(t)$ if the noise

processes $\{N_\alpha(t), N_\beta(t)\}$ are independent of $\{N_i(t): i \neq \alpha, \beta\}$ and the probabilities of $N_\alpha(t) = 1$ and of $N_\beta(t) = 1$ are equal.

ERROR FOR INDEPENDENT AND STATIONARY INPUTS

The noise-error formulas in Eqs. (25) and (26) are quite involved, and it is difficult to interpret the dependence of $\xi_S(N)$ and $\xi_S(N, X)$ on the statistics of the noise and the signal. We shall now show that when the signal and noise are independent and stationary processes, each identically distributed, $\xi_S(N)$ is expressible as a polynomial in the expected value of the noise and $\xi_S(N, X)$ as a polynomial in the expected values of the signal and the noise. Preparatory to the substantiation of this statement, we first need to establish a lemma.

Lemma (See Appendix II for proof)

If $X_j \in \{0, 1\}$ are independent and identically distributed binary random variables ($0 \leq j < n$) with Prob. $(X_j = 1) = \delta_X$, then

$$E\{Wal(i, \bar{X})\} = (1 - 2\delta_X)^{Hm(i)} \quad (29)$$

where $Hm(i)$ denotes the Hamming weight of i :

$$Hm(i) = \sum_{j=0}^{n-1} i_j \quad (30)$$

Under the assumption that the signal and noise are independent and stationary processes, each identically distributed, we obtain the following important simplified results from Eqs. (25) and (26) immediately, with the aid of Eq. (29).

Assertion III

If $\{X_i(t), N_i(t): 0 \leq i < n\}$ are independent and stationary with

$$\text{Prob}[X_i(t) = 1] = E\{X_i\} = \delta_X \quad (31)$$

and

$$\text{Prob}[N_i(t) = 1] = E\{N_i\} = \delta_N \quad (32)$$

then the component error expressions $\xi_S(N)$ and $\xi_S(N,X)$ in Eqs. (25) and (26) for the Boolean function S represented in Eq. (10) can be simplified respectively to:

$$\xi_S(N) = 2 \sum_{i \in D_n} b_i^2 [1 - p_N^{\text{Hm}(i)}] \quad (33)$$

and

$$\xi_S(N,X) = \sum_{i \neq j: i, j \in D_n} b_i b_j [1 + p_N^{\text{Hm}(i) \oplus j} - p_N^{\text{Hm}(i)} - p_N^{\text{Hm}(j)}] p_X^{\text{Hm}(i) \oplus j} \quad (34)$$

where

$$p_X = 1 - 2\delta_X \quad (35)$$

and

$$p_N = 1 - 2\delta_N \quad (36)$$

and D_n may be replaced by $D_n - \{0\}$. Note that $\xi_S(N)$ in Eq. (33) is a polynomial in the expected value of the noise and that $\xi_S(N,X)$ in Eq. (34) is a polynomial in the expected values of the signal and the noise. Both are relatively simple to compute and their sum can be plotted versus δ_N for different values of δ_X for a given combinational circuit.

SPECIAL SITUATIONS

We now examine the behavior of the error for three special situations; namely, (A) the low-noise case, (B) the case of $\delta_N = 0.5$, and (C) the unbiased-signal case.

(A) The low-noise case: $\delta_N \ll 1$.

In such a case, we can use the approximation

$$p_N^k = (1 - 2\delta_N)^k \approx 1 - 2k\delta_N \quad (37)$$

and write Eqs. (33) and (34) as

$$\xi_S(N) = 2\delta_N e_0 \quad (38)$$

and

$$\xi_S(N, X) = 2\delta_N \sum_{k=1}^n e_k p_X^k \quad (39)$$

where, for $0 \leq k \leq n$

$$e_k = \sum_{i,j \in D_n} b_i b_j [Hm(i) + Hm(j) - Hm(i \oplus j)] \quad (40)$$

$$Hm(i \oplus j) = k$$

Substituting Eqs. (38) and (39) in Eq. (24), we have

$$\xi_S = 2\delta_N \sum_{k=0}^n e_k p_X^k \quad (41)$$

Hence we can conclude from Eq. (41) that in a low-noise situation the error ξ_S increases approximately linearly with δ_N , the constant of proportionality being a polynomial in p_X . This conclusion is exhibited in Figs. 2(a) and 3(a). Note that ξ_S is independent of the signal if and only if $e_k = 0$ for $0 < k \leq n$.

(B) The case of $\delta_N = 0.5$.

This is a case of very high noise and, from Eq. (36), $p_N = 0$.

We have

$$1 + p_N^{Hm(i \oplus j)} - p_N^{Hm(i)} - p_N^{Hm(j)} = \begin{cases} 2, & \text{for } i = j \neq 0 \\ 1, & \text{for } i \neq j \neq 0 \\ 0, & \text{for } i \text{ or } j \text{ or both} = 0 \end{cases} \quad (42)$$

Substitution of Eq. (42) in Eq. (34) and then combining with Eq. (33), we obtain

$$\xi_S = \xi_S(N) + \xi_S(N, X) = \sum_{k=0}^n c_k p_X^k \quad (43)$$

where

$$c_0 = 2 \sum_{i \in D_n - \{0\}} b_i^2 \quad (= \xi_S(N)) \quad (44)$$

and

$$c_k = \sum_{i, j \in D_n - \{0\}} b_i b_j, \quad 1 \leq k \leq n \quad (45)$$

$$Hm(i \oplus j) = k$$

Hence ξ_S is a polynomial in p_X as in Eq. (41). ξ_S is independent of signal $\bar{X}(t)$ if and only if $c_k = 0$ for $1 \leq k \leq n$.

(C) The unbiased-signal case

In many situations the signal has no bias, or the environment is not known; it would be reasonable to assume $\delta_X = 0.5$ or $p_X = 0$. Hence $\xi_S(N, X) = 0$ and

$$\xi_S = \xi_S(N) = 2 \sum_{i \in D_n} b_i^2 [1 - p_N^{Hm(i)}] \quad (46)$$

The error is then invariant over each equivalence class.

NUMERICAL EXAMPLES AND ERROR CURVES

We shall now apply the method developed in the previous sections to determine the error at the output of two combinational circuits as a function of the expected values of the signal and the noise at the input. Both the signal and the noise component stochastic processes are assumed to be independent and stationary, each being identically distributed. The error ξ_S will be calculated and plotted versus δ_N for different

values of δ_X . Two sets of error curves (one set for low-noise and the other set for high-noise situation) are presented for the Boolean function of each combinational circuit.

Circuit 1 - A 3-input function: $S(\bar{x}) = x_2'x_1x_0' + x_2x_1'x_0' + x_2x_1$.

For this Boolean function, $S(\bar{x}) = 1$ for $x \in \{2, 4, 6, 7\}$ and the vector \bar{b} representing the coefficients b_i ($i = 0, 1, \dots, 7$) of the Walsh-series expansion in Eq. (10) is found by Eq. (11) to be

$$\bar{b} = 2^{-2} [2 \ 1 \ -1 \ 0 \ -1 \ 0 \ 0 \ -1]$$

In order to examine the behavior of the error curves in both the low-noise and the high-noise situations, the e_k coefficients in Eq. (40) and the c_k coefficients in Eqs. (44) and (45) are computed.

$$e_0 = 0.75, \quad e_1 = e_3 = 0, \quad e_2 = 0.25$$

$$c_0 = 0.50, \quad c_1 = c_2 = c_3 = 0$$

ξ_S vs. δ_N curves are plotted in Figs. 2(a) and 2(b). It is seen that in the low-noise range ξ_S increases almost linearly with δ_N and is dependent on δ_X because e_2 is non-zero. In the entire high-noise range, ξ_S depends minimally on δ_X and becomes independent of δ_X at $\delta_N = 0.5$, in agreement with the c_k 's being zero for $k \neq 0$. The error curves are monotonically increasing, and thus a maximum uncertainty ($\delta_N = 0.5$) does not cause a maximum error. The compactness of both the low-noise and the high-noise error curves is apparently due to the sparseness of the vector \bar{b} .

Circuit 2 - A 4-input function $S(\bar{x}) = x_3'x_2'(x_1 \oplus x_0) + (x_3 \oplus x_2)x_1'x_0' + (x_3' + x_3x_2')x_1x_0$.

Here, $S(\bar{x})=1$ for $x \in \{1,2,3,4,7,8,11\}$, and the vector \bar{b} is

$$\bar{b} = 2^{-4}[7 \quad -1 \quad -1 \quad 3 \quad 3 \quad -1 \quad -1 \quad -1 \quad 3 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -5]$$

and the e_k and c_k coefficients are:

$$e_0 = 1.250, \quad e_1 = 0.750, \quad e_2 = 0.250, \quad e_3 = -0.250, \quad e_4 = 0$$

$$c_0 = 0.492, \quad c_1 = 0.0312, \quad c_2 = -0.0156, \quad c_3 = -0.0312, \quad c_4 = -0.0391$$

ξ_S vs. δ_N curves are plotted in Figs. 3(a) and 3(b). In the low-noise range, ξ_S increases almost linearly with δ_N . The slope of the error curves decreases fairly linearly with increasing δ_X ; hence the low-noise error is upper-bounded by the $\delta_X = 0$ line. This linear dependence on δ_X results from the dominance of e_1 over the other e_k 's. The high-noise curves demonstrate two phenomena: firstly, heavily biased signals ($\delta_X = 0.001, 0.01, 0.90, 0.99$) suffer less error with greater noise above $\delta_N = 0.5$; secondly, the value of δ_N at which the maximum error occurs increases with increasing δ_X . This complicated behavior is a hallmark of the absence of, or having few, zero Walsh coefficients (b_i 's).

CONCLUSION

This report presents a procedure for analyzing the stochastic behavior of digital circuits by the use of Walsh functions. In particular, the error at the output of a combinational circuit caused by noise is studied by defining a distance measure between the responses to the signal and to the noise-corrupted signal. Restricting the noise to be dyadically additive, which is perfectly reasonable, Walsh representation is used to obtain the error in terms of the input statistics. It is shown that the error can be

expressed as the sum of two parts: one part depends only on noise statistics, and the other on both signal and noise. The former is invariant for equivalent Boolean functions if the noise processes are independent and identically distributed. Under the more constrained condition of independent and stationary noise and signal processes, each identically distributed, the error is a polynomial function of the expected values of signal and of noise. In low-noise situation, the error increases linearly with the expected value of the noise at the input. For unbiased signals the error polynomial is invariant over each equivalence class. These properties are exhibited in two typical examples.

APPENDICES

(I) Proof of Assertion I

(a) By hypothesis,

$$Q(\bar{x}) = 1 - S(\bar{x}) \quad (\text{A-1})$$

Using Eq. (10) and the fact that $\text{Wal}(0, \bar{x}) \equiv 1$, we have

$$\begin{aligned} Q(\bar{x}) &= \text{Wal}(0, \bar{x}) - \sum_{i \in D_n} b_i \text{Wal}(i, \bar{x}) \\ &= (1 - b_0) \text{Wal}(0, \bar{x}) - \sum_{i \in D_n - \{0\}} b_i \text{Wal}(i, \bar{x}) \end{aligned} \quad (\text{A-2})$$

Comparison of Eq. (A-2) with Eq. (13) proves Eq. (14).

(b) Complementing the m inputs changes \bar{x} to $\bar{y} = \bar{x} \oplus \bar{h}$ with h given by Eq. (15a) which implies $h_{j_k} = 1$ for all k . As a result,

$$\begin{aligned}
Q(\bar{x}) &= S(\bar{y}) = S(\bar{x} \oplus \bar{h}) \\
&= \sum_{i \in D_n} b_i \text{Wal}(i, \bar{x} \oplus \bar{h}) \\
&= \sum_{i \in D_n} [b_i \text{Wal}(i, \bar{h})] \text{Wal}(i, \bar{x})
\end{aligned} \tag{A-3}$$

from which Eq. (15) follows directly.

(c) Interchanging x_α and x_β changes \bar{x} to \bar{y} and

$$Q(\bar{x}) = S(\bar{y}) = \sum_{i_\alpha = i_\beta} b_i \text{Wal}(i, \bar{y}) + \sum_{i_\alpha \neq i_\beta} b_i \text{Wal}(i, \bar{y}) \tag{A-4}$$

(c-1) If $i_\alpha = i_\beta$, $\text{Wal}(i, \bar{y}) = \text{Wal}(i, \bar{x})$. Thus, $q_i = b_i$.

(c-2) If $i_\alpha \neq i_\beta$, then for $j = i \oplus 2^\alpha \oplus 2^\beta$, $j_\alpha \neq j_\beta$. We have

$$\text{Wal}(i, \bar{y}) = \text{Wal}(j, \bar{x}), \text{ and } \text{Wal}(j, \bar{y}) = \text{Wal}(i, \bar{x}).$$

$$\text{Thus, } q_i = b_i \oplus 2^\alpha \oplus 2^\beta.$$

Equation (16) is therefore proved.

(II) Proof of Lemma

We note from Eq. (6) that only those x_j 's corresponding to $i_j = 1$ will have an effect on the value of $\text{Wal}(i, \bar{x})$. If the X_j 's are independent and identically distributed, any collection of that many X_j 's will suffice for probability calculations for $\text{Wal}(i, \bar{x})$. Thus,

$$\begin{aligned}
\text{Prob} [\text{Wal}(i, \bar{x}) = 1] &= \text{Prob} \left[\left(\sum_{j=1}^{Hm(i)} X_j \right) \text{ is even} \right] \\
&= \sum_{w=0}^{Hm(i)} \text{Prob} \left[\left(\sum_{j=1}^{Hm(i)} X_j \right) = w \right] \\
&\quad (w \text{ is even}) \\
&= \sum_{w=0}^{Hm(i)} \binom{Hm(i)}{w} \delta_X^w (1 - \delta_X)^{Hm(i)-w} \\
&\quad (w \text{ is even})
\end{aligned} \tag{A-5}$$

Similarly,

$$\begin{aligned}
 \text{Prob} [\text{Wal}(i, \bar{X}) = -1] &= \text{Prob} \left[\left(\sum_{j=1}^{\text{Hm}(i)} X_j \right) \text{ is odd} \right] \\
 &= \sum_{\substack{w=1 \\ (w \text{ is odd})}}^{\text{Hm}(i)} \binom{\text{Hm}(i)}{w} \delta_X^w (1-\delta_X)^{\text{Hm}(i)-w}
 \end{aligned} \tag{A-6}$$

Combining Eqs. (A-5) and (A-6), we have

$$\begin{aligned}
 E[\text{Wal}(i, \bar{X})] &= \text{Prob} [\text{Wal}(i, \bar{X}) = 1] - \text{Prob} [\text{Wal}(i, \bar{X}) = -1] \\
 &= \sum_{w=0}^{\text{Hm}(i)} \binom{\text{Hm}(i)}{w} (-\delta_X)^w (1-\delta_X)^{\text{Hm}(i)-w} \\
 &= (1 - 2\delta_X)^{\text{Hm}(i)}
 \end{aligned} \tag{A-7}$$

which is Eq. (29), the lemma to be proved.

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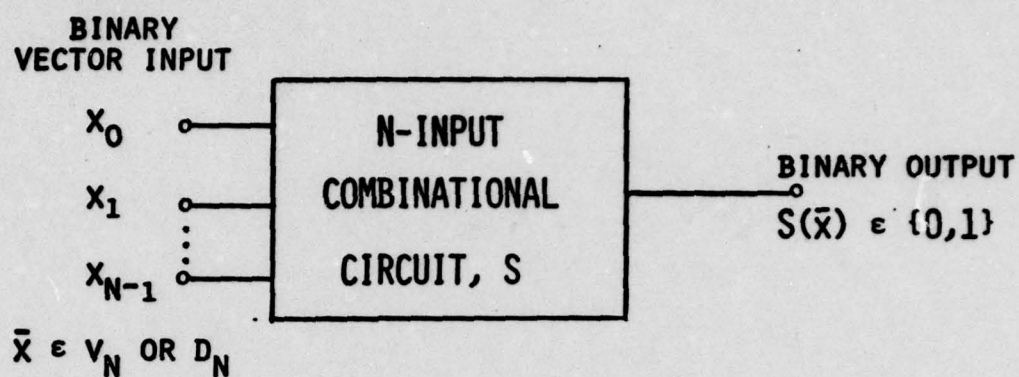


Fig. 1 - Schematic diagram of an n-input combinational circuit.

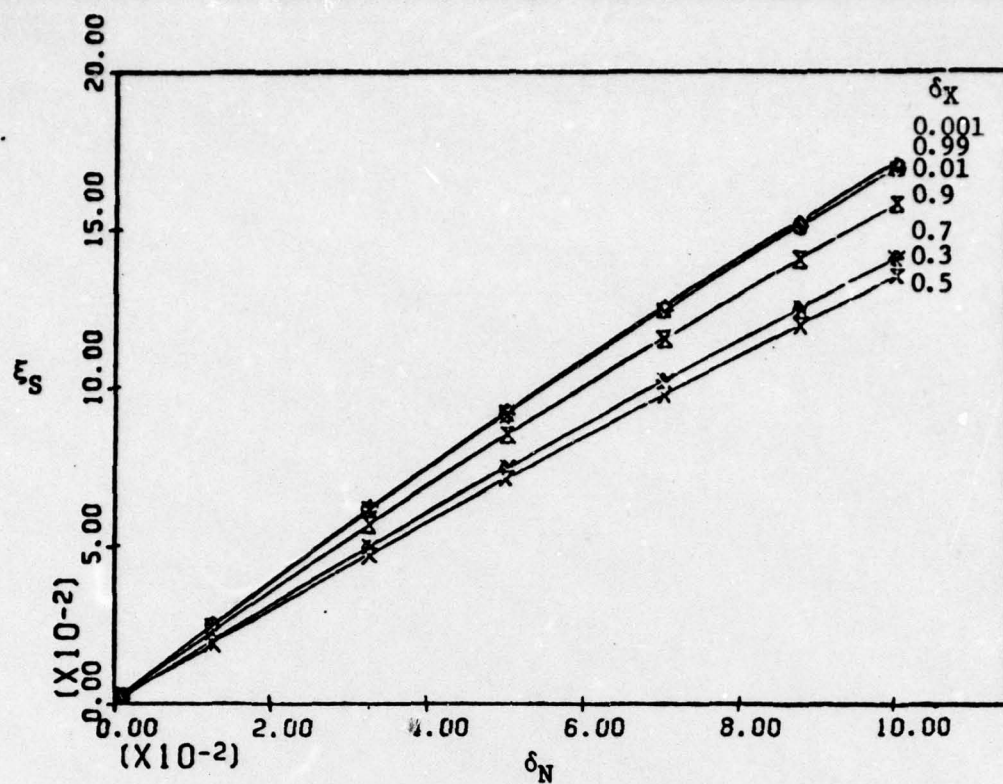


Fig. 2(a). Low-noise error curve for circuit 1.

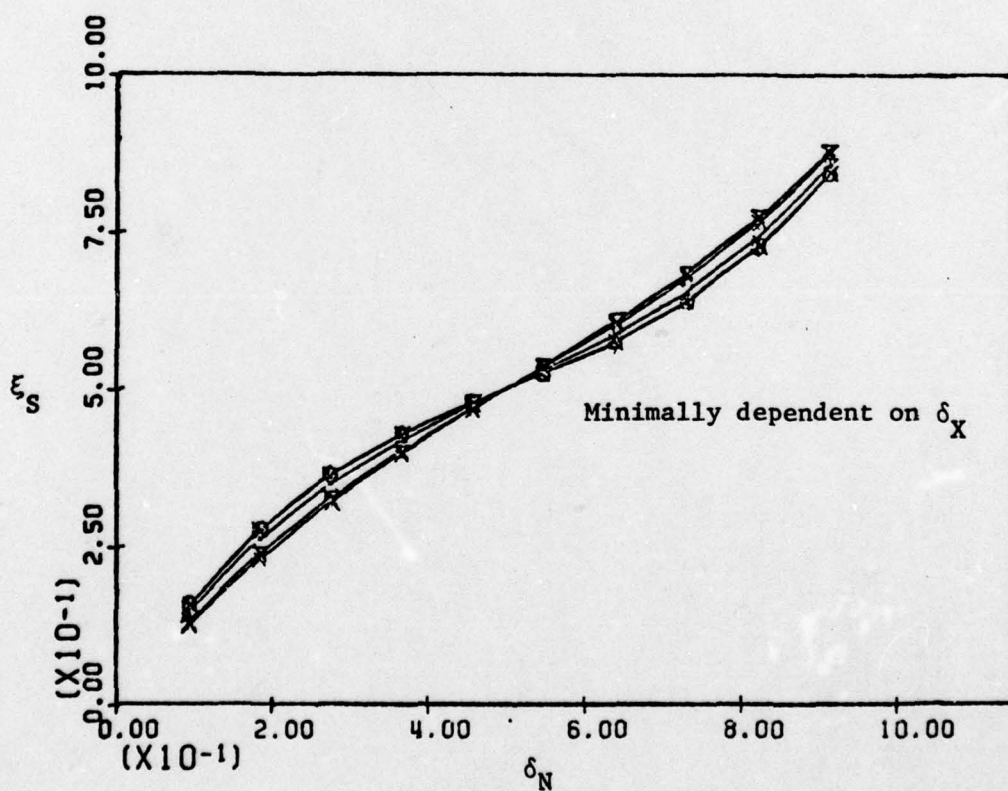


Fig. 2(b). High-noise error curve for circuit 1

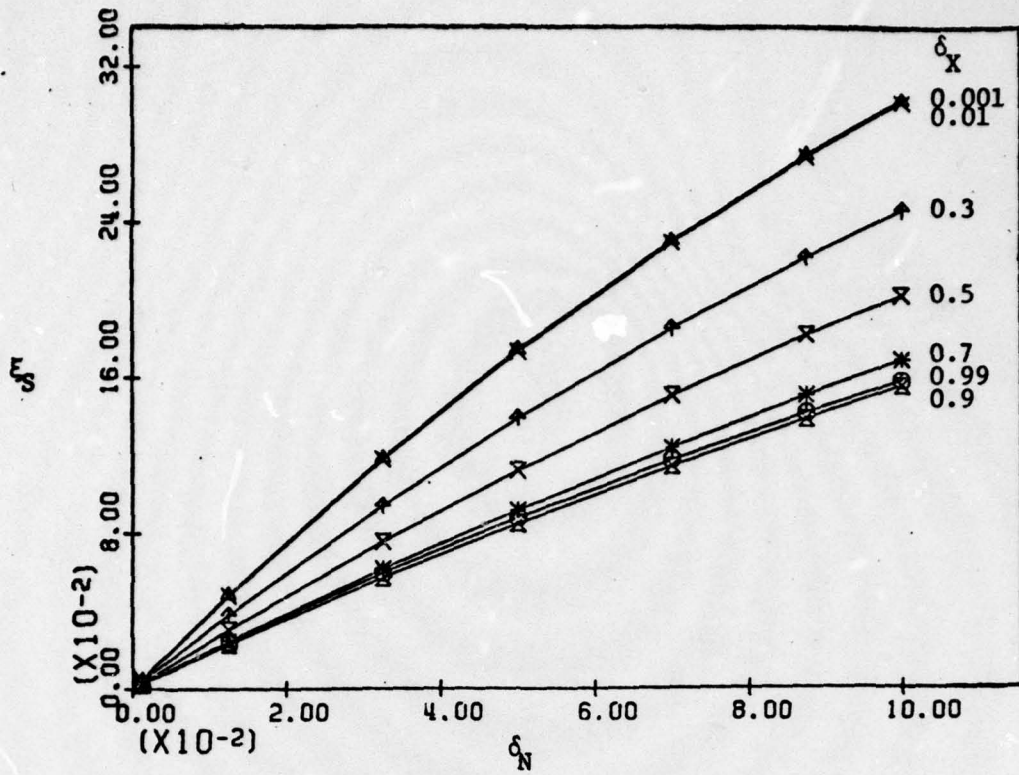


Fig. 3(a). Low-noise error curve for circuit 2.

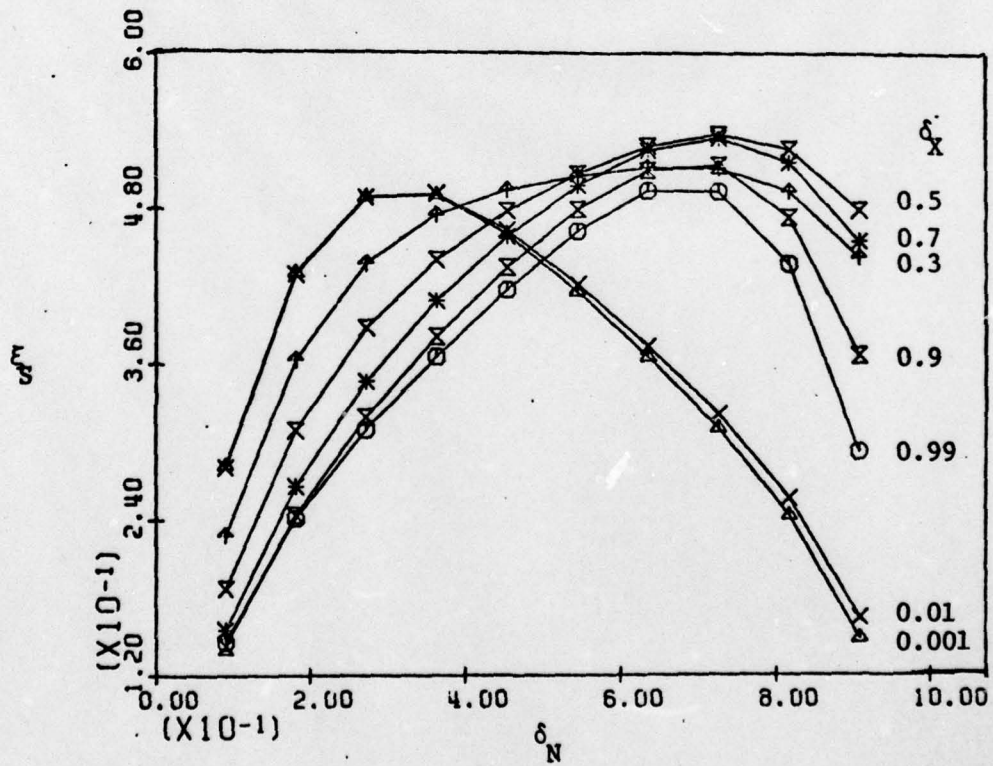


Fig. 3(b). High-noise error curve for circuit 2.